VOLUME 76 SEPARATE No. D-5

Rochester Public Library
115 South Avenue
Rochester 4, N.Y.

PROCEEDINGS

AMERICAN SOCIETY OF CIVIL ENGINEERS

November, 1950

DISCUSSION OF ELASTIC RESTRAINT EQUATIONS FOR SEMI-RIGID CONNECTIONS

(*Published in February, 1950*)

By Lloyd T. Cheney, Thomas C. Kavanagh, Dean F. Peterson, Jr., and Jack E. Cermak, and J. E. Lothers

STRUCTURAL DIVISION

Copyright 1950 by the AMERICAN SOCIETY OF CIVIL ENGINEERS
Printed in the United States of America

Headquarters of the Society
33 W. 39th St.
New York 18, N.Y.

PRICE $0.50 PER COPY

The Society is not responsible for any statement made or opinion expressed in its publications

---

Published at Prince and Lemon Streets, Lancaster, Pa., by the American Society of Civil Engineers. Editorial and General Offices at 33 West Thirty-ninth Street, New York 18, N. Y. Reprints from this publication may be made on condition that the full title of paper, name of author, page reference, and date of publication by the Society are given.

DISCUSSION

LLOYD T. CHENEY,[13] JUN. ASCE.—An interesting analytical approach to the problem of evaluating the rigidity of standard beam web-angle connections is presented in this paper. As stated by the author, several factors which contribute to the rotation of the connection have been neglected in the derivation of Eq. 22.

It should be noted that the comparison drawn in Fig. 4 between the experimental results of J. Charles Rathbun,[14] M. ASCE, and the slopes of the moment-rotation curves as computed by Eq. 22 can be valid only when the beam is framed to the web of a column or girder rather than to the column flanges as illustrated in Fig. 3(*a*). When the tension rivets of a web-angle connection, a top-angle and seat-angle connection, or a tee and seat-angle connection are joined with the column flanges, it is not reasonable to neglect the deformations of the column flanges. L. E. Grinter, M. ASCE, in his discussion[15] of Mr. Rathbun's paper, observed, among the data therein presented, the large contribution to the rotation of the joint that was made by "* * * the duplication of tension rivets and the deformation of the column section." Additional evidence of this column-flange distortion was observed by the writer during a program to investigate the effect of various types of building connections on the local buckling of column flanges.[16]

The question arises as to why the author has developed the analysis for these beam web-angle connections. This type of connection, of course, does offer some moment resistance but it is relatively inefficient in that respect when compared with the moment resistance obtainable with the same number of rivets in top-angle and seat-angle connections. If the engineer wishes to obtain economy in design by taking advantage of the end restraint of a connection, he would do well to employ one of the more efficient moment resistant types of connection. Robert A. Hechtman, Assoc. M. ASCE, and Bruce G. Johnston, M. ASCE, in their investigation of semi-rigid beam to column connections,[17] found the web-angle connections relatively inefficient and concluded that:

> "It was evident from the first series of web angle tests that the top and seat angle connection offered greater end restraint effectiveness and the remainder of the program was devoted to this type connection * * *."

NOTE.—This paper by J. E. Lothers was published in February, 1950. The numbering of footnotes, equations, tables, and illustrations in this Separate is a continuation of the consecutive numbering used in the original paper.

[13] Asst. Prof., Civ. Eng., Wayne Univ., Detroit, Mich.

[14] "Elastic Properties of Riveted Connections," by J. Charles Rathbun, *Transactions*, ASCE, Vol. 101, 1936, pp. 527 and 528.

[15] *Ibid.*, p. 571.

[16] "Steel Columns of Rolled Wide Flange Section," by Bruce G. Johnston and Lloyd A. Cheney, *Progress Report No. 1*, Committee on Steel Structures Research, AISC, 1942, p. 31.

[17] "Riveted Semi-Rigid Beam-to-Column Building Connections," by Robert A. Hechtman and Bruce G. Johnston, *Progress Report No. 1*, Committee on Steel Structures Research, AISC, 1947, p. 39.

The summary of their test results shows very strikingly the justification of this conclusion. Messrs. Hechtman and Johnston report[18] that a connection very similar to the standard KK-4 has a percentage of rigidity of 12.6 for uniform load and a load factor of 1.65 whereas a top-angle and seat-angle connection with the same number of rivets and a slightly heavier top angle has a percentage of rigidity of 31.4 under the same conditions of load. As a result of that investigation, Messrs. Hechtman and Johnston tabulated the dependable percentage of rigidity for a series of proposed standard semi-rigid connections. It may well be that in the near future these proposed connections and their experimentally determined constants will be accepted as standard.

In his concluding remarks Mr. Lothers comments that the same reasoning employed in the derivation of Eqs. 18 through 35 for the web-angle connections may be applied to the split-beam connection, the top-angle and the seat-angle connections, etc., and that such application would constitute an improvement over the present method of laboratory tests. A connection tested by Messrs. Hechtman and Johnston is believed to be typical.[19] Since the factors considered by the author as contributing to the rotation of the connection do not involve the largest part of the total rotation as assumed, it is apparent that to ignore the contribution of the other elements will lead to results quite different from actual connection performance.

THOMAS C. KAVANAGH,[20] ASSOC. M. ASCE.—The development, theoretically, of basic data required to realize the advantages of semi-rigid connections is a worthwhile contribution to the literature.

The need for data of this kind is evidenced in a paper by Leroy A. Beaufoy, Assoc. M. ASCE, and A. Moharram,[21] wherein the curves of M versus ϕ for standard web connections of the type used by the author have been derived from experimental data on joints connected at the flanges. Perhaps the most obvious observation in connection with this latter paper is the concern of these British investigators with the complete curve of M versus ϕ. The same was true of the Final Report of the Steel Structures Research Committee[4] wherein a general curve of the form $M = C\,\phi^{0.412}$ was proposed for several of these connections.

The paper by Mr. Lothers, in comparison, yields the initial tangent apparently with good experimental agreement. The question arises as to how adequate this initial value is for practical work. Actually, no part of the moment curve shows a linear relationship, and permanent set takes place on unloading even at low moments, which is evidence of plastic deformation. The initial tangent is then an upper limit of the stiffness, and this must be taken into account in designing with it on the semi-rigid theory.

---

[18] "Riveted Semi-Rigid Beam-to-Column Building Connections," by Robert A. Hechtman and Bruce G. Johnston, *Progress Report No. 1*, Committee on Steel Structures Research, AISC, 1947, pp. 32 and 33.

[19] *Ibid.*, p. 99, test No. 20.

[20] Prof. of Civ. Eng., Pennsylvania State College, State College, Pa.

[21] "Derived Moment-Angle Curves for Web-Cleat Connections," by Leroy A. Beaufoy and A. Moharram, Preliminary Publication of the 3d Cong., International Assn. for Bridge and Structural Eng., Liege, Belgium, 1948.

[4] "First, Second, and Final Reports," Steel Structures Research Committee, Dept. of Scientific and Industrial Research of Great Britain, H. M. Stationery Office, London, 1931–1936.

The writer would like to question whether the fact that the connection angles are never in perfect contact with the column will not favor the assumption of the neutral axis at the center of the connection for low values of the moment, as compared with the author's procedure. Messrs. Beaufoy and Moharram [21] refer to tests showing horizontal movements on the compression side of the connection, corresponding to a "closing-in" of the initial clearance existing between the connection angle and the face of the support. Initially, this "closing-in" is almost as large as the"opening-up" on the tension side, but it disappears with larger application of moment. This is of interest, particularly because it is precisely in the low moment region that the best experimental agreement occurs.

Table 1 should be of considerable use to designers of semi-rigid structures. It is quite probable, however, that the moment M indicated therein will be higher in many practical cases than will be allowed by combined moment and shear on the standard number of rivets usually provided in these connections.

DEAN F. PETERSON, JR.,[22] ASSOC. M. ASCE, AND JACK E. CERMAK,[23] JUN. ASCE.—There can be no doubt that one of the major uncertainties in the stress analysis of building frames arises from the analyst's inability to evaluate the modifying effects of the connection on the end restraints of the members. Since 1940, fairly simple mathematical methods have been developed for introducing the effects of the strains in the connections into the analysis, and answers that consider the effects of these strains may be quite readily obtained [2,24] as far as the mathematics of analysis are concerned. The practical difficulty, however, has been that the specific elastic behavior (ratio of angle change to moment, Z) of the joint connection must be ascertained by experiment. The author has presented formulas for predicting such behavior on the basis of the physical dimensions of the angles forming the connection for a certain type of connection.

In considering the case of a beam connected to a column by a pair of vertical angles riveted to the web of the beam, the author deduces (using the slope deflection equations) an expression for the horizontal deflection, Δ_b, at the top of the connection. This deflection is expressed as a function of the pull (P) per unit vertical length of connection at the top of the connecting angles (Eq. 14). The connection is then considered analogous to a section in flexure. The angle or rotation (ϕ) and the bending moment (M) are expressed in terms of the bending stress (s) in the connecting angle, the distance (y) to the neutral axis (Fig. 3) and the physical dimensions of the angles forming the connections (see Eqs. 17 and 18.) To evaluate M and ϕ one must know s and y. Both values are statically indeterminate and some additional assumptions must be made in order to find them. In the expression for Z (Eq. 22) the value of s,

[22] Prof. and Head of Civ. Eng., Colorado Agri. and Mech. College, Fort Collins, Colo.

[23] Asst. Prof. of Civ. Eng., Colorado Agri. and Mech. College, Fort Collins, Colo.

[2] "Analysis of Building Frames with Semi-Rigid Connections," by Bruce Johnston and Edward H Mount, *Transactions*, ASCE, Vol. 107, 1942, p. 993.

[24] "Analysis of Frames with Elastic Joints," by Ralph W. Stewart, *ibid.*, Vol. 114, 1949, p. 17.

however, disappears so that only y must be determined. The method for determining y is discussed under the heading, "Analysis: The Neutral Axis."

The writers experienced some difficulty in following the reasoning of the author in deriving the expression for the position of the netural axis. The key to this derivation appears to be in the statement (four lines below Eq. 19):

> "It is assumed that the ratio of bending stress to strain in the column-connected leg of the angle along rivet line A is the same as that in compression below the neutral axis."

The writers infer that this statement, in mathematical terms, means

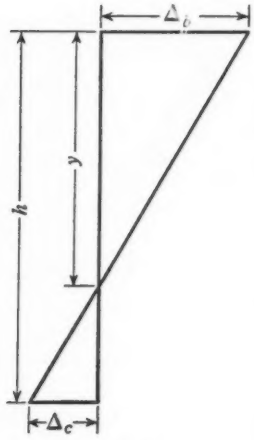

FIG. 6

$$\frac{s}{\Delta_b} = \frac{s_c}{\Delta_c} \quad \ldots\ldots\ldots\ldots\ldots (36)$$

in which s is the maximum bending stress in the angle; s_c is the maximum compressive stress between the column-connected angle leg and the column; and Δ_c is the corresponding strain. From Fig. 6,

$$\Delta_b = \frac{y}{h - y}\,\Delta_c \quad \ldots\ldots\ldots\ldots\ldots (37)$$

Substituting from Eq. 36,

$$s = \frac{y}{h - y}\,s_c \quad \ldots\ldots\ldots\ldots\ldots (38)$$

Eq. 38 states that the compressive stress at the column-connected leg is in proportion to the maximum bending stress at rivet line A as the distance from the neutral axis. As stated by the author, the total shearing force across the rivet line Section A is opposed by the total compressive force below the neutral axis on the column-connected leg and, from the condition that these forces are equal,

$$\frac{v\,t\,y}{2} = \frac{s_c\,b\,(h - y)}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (39)$$

Substituting from Eq. 38,

$$\frac{v\,t\,y^2}{2} = \frac{s\,b\,(h - y)^2}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (40a)$$

Substituting the value of s in terms of v (Eq. 19) and the value of n (Eq. 20),

$$\frac{t\,y^2}{2} = \frac{n\,b\,(h - y)^2}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (40b)$$

which leads to Eq. 21 for y. The foregoing discussion is presented to help clarify the author's discussion leading to Eq. 21.

The writers take issue with the major premise on which the derivation of Eq. 21 is based—that is, that $\Delta_b/s = \Delta_c/s_c$. Although Δ_b is probably propor-

tional to s, and although Δ_c is probably proportional to s_c, there is no justification for assuming that these proportions are equal. The writers would normally expect Δ_b/s to be much larger than Δ_c/s_c. It also appears illogical that the relationship between Δ_b/s and Δ_c/s_c is independent of the dimensions of the connections.

If one considers that the modulus of elasticity in bending equals the modulus in compression, an expression for y may be deduced which is not subject to the foregoing criticism. Substituting the value of P from Eq. 16 in Eq. 14, remembering $I = t^3/12$, and solving for E_b,

$$E_b = \frac{2}{3}\frac{s\,g^2}{\Delta_b\,t}\left(\frac{g+g_1}{2\,g+g_1}\right) \ldots\ldots\ldots\ldots\ldots\ldots (41)$$

Most of the strain on the compression side occurs as the result of compression of the beam-connected leg between the rivet holes and the column, and the unit compressive strain at the extreme fiber is $\frac{\Delta_c}{g_1+t}$. The modulus of elasticity in compression is expressed by

$$E_c = \frac{s'_c}{\Delta_c}(g_1+t) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (42a)$$

in which s'_c is the maximum compressive stress in the beam-connected leg of the connecting angle.

Substituting the value of Δ_c from Eq. 37 yields

$$E_c = \frac{s'_c}{\Delta_b}\left(\frac{y}{h-y}\right)(g_1+t) \ldots\ldots\ldots\ldots\ldots\ldots (42b)$$

Equating Eqs. 41 and 42*b* yields

$$s = \frac{3}{2}s'_c\left(\frac{y}{h-y}\right)\left(\frac{2\,g+g_1}{g+g_1}\right)(g_1+t)\,\frac{t}{g^2} \ldots\ldots\ldots\ldots (43)$$

Substituting s from Eq. 43 into Eq. 16,

$$v = \frac{s'_c}{4}\left(\frac{y}{h-y}\right)\left(\frac{4\,g+g_1}{g+g_1}\right)(g_1+t)\,\frac{t^2}{g^3} \ldots\ldots\ldots\ldots (44)$$

Let—

$$n' = \frac{4\,g^3}{t^2\,(g_1+t)}\left(\frac{g+g_1}{4\,g+g_1}\right) \ldots\ldots\ldots\ldots\ldots\ldots (45)$$

it follows that—

$$v = \frac{s'_c}{n'}\left(\frac{y}{h-y}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (46)$$

Since the total shear on the section through A must equal the compression in the column-connected leg below the neutral axis,

$$\frac{s'_c\,t\,y}{2\,n'}\left(\frac{y}{h-y}\right) = \frac{s'_c}{2}(h-y)\,t \ldots\ldots\ldots\ldots\ldots\ldots (47)$$

from which

$$y = \frac{\sqrt{n'}}{1 - \sqrt{n'}} h \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (48)$$

Eq. 48 is based on the proved physical property of steel, that its compression modulus is equal, or nearly so, to its bending modulus. Eq. 48, therefore, is considered to be superior to Eq. 21, which appears to be based on a rather arbitrary, if not questionable, assumption. In the development of a relationship between physical quantities, the use of any special assumption applying only to this case, indicates that either the basic principles of mechanics are not correct and are in need of restatement, or the development has been oversimplified.

The formula for Z may be simplified by substituting the value for n' into Eq. 22 to

$$Z = \frac{3}{2} \frac{g_1 + t}{E\, h\, t\, y^2} n' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (49)$$

The values of $1/Z$ from Eq. 49 have been computed for several different sizes of connections and are listed in Tables 3 and 4 with the corresponding values

TABLE 3.—COMPARISON OF CONSTANTS FOR STANDARD CONNECTIONS, AMERICAN INSTITUTE OF STEEL CONSTRUCTION

Connection (see Table 1) (1)	Beam web thickness (2)	$\frac{1}{Z \times 10^6}$ (Eq. 22) (3)	$\frac{1}{Z \times 10^6}$ (Eq. 49) (4)
A(a)	$\frac{1}{4}$	0.422	0.345
A(a)	$\frac{3}{8}$	0.319	0.281
A(c)	$\frac{1}{4}$	3.692	3.040
A(c)	$\frac{3}{8}$	2.888	2.455

TABLE 4.—COMPARISON OF CONSTANTS FOR PUBLISHED LABORATORY RESULTS

Specimen (1)	$\frac{1}{Z \times 10^6}$ (Eq. 22) (2)	$\frac{1}{Z \times 10^6}$ (Eq. 49) (3)
4	0.407	0.357
5	0.577	0.477
6	2.037	1.775
7	2.906	2.370

obtained by the author. In all instances the values of $1/Z$ computed by Eq. 49 are definitely lower than those computed by Eq. 22.

The author is to be commended for introducing a rational approach to the problem of evaluating the stiffness factors for standard connections. Introduction of this concept into structural analysis of building frames will lead to more certainty about stress conditions and perhaps greater economy in design.

J. E. LOTHERS,[25] M. ASCE.—The extent of the discussion by Messrs. Peterson and Cermak is sincerely appreciated. They carry an analysis of their own through to a final expression for Z. Also, grateful thanks are due Messrs. Cheney and Kavanagh who find the paper worthy of some careful thought, and both of whom raise some thought-provoking questions.

A comparison of the $\frac{1}{Z}$-values computed by Eq. 49 with those computed by Eq. 22 (see Tables 3 and 4) indicates that the slopes from the former are

[25] Prof. of Architecture, Oklahoma Agri. and Mech. College, Stillwater, Okla.

less steep; and, if plotted on the moment-rotation curves[26] proposed by Mr. Rathbun for his specimens 5 to 7, inclusive, they will fall tangent to those curves at points considerably above the origin. In the case of specimen 7, for example, the slope as computed from Eq. 49 falls tangent approximately at the 150-in-kip level. The resisting moment of the connection used in specimen 7, as computed by Eq. 31, is only 66.4 in-kips. In other words, to use Eq. 49 for design purposes in the case of specimen 7 is equivalent to permitting a bending stress of 45.3 kips per sq in. in the connection angles. Eq. 22, on the other hand, yields a slope that is tangent at approximately the 66.4-in-kip level which corresponds to a design stress of 20 kips per sq in.

The writer does not mean to disparage the value of Eq. 49, however. As a matter of fact, it yields slopes that are more nearly in keeping with present design practice than does Eq. 22. In fact, a stress of 45 kips per sq in. in the connection is not considered serious. Attention is called to an illustration[27] presented by Bruce Johnston, M. ASCE, and Edward H. Mount which indicates their recommended design range for $\frac{1}{Z}$-values. It will be noted that slopes from both Eqs. 22 and 49 fall within that range.

Mr. Cheney questions the writer's judgment in selecting the web-angle type of connection for his analyses. There were two reasons for so doing: (1) The web-angle connection has been more or less standardized and, therefore, is a very commonly used connection; and (2) there is only one other type of connection for which elastic restraint equations would be justified—namely, the combined web-angle and seat-and-clip connection. The purely seat-and-clip-angle type of connection should be avoided. It is the least desirable of all the commonly used beam connections. The writer wishes to take exception to Mr. Cheney's statement to the effect that, with the same number of rivets, the top-angle and seat-angle connection offers more moment resistance than the web-angle connection. Attention is again called to the Rathbun moment-rotation curves, this time to specimens 5, 8, and 9.[28] Specimen 5 is a web-angle connection requiring twenty-four rivets. At a rotation of 0.001 radian this connection developed a resisting moment of 210 in-kips. Specimen 8 is what Mr. Cheney refers to as a top-angle and seat-angle connection. This specimen requires twenty-two rivets and at a rotation of 0.001 radian developed a resisting moment of 140 in-kips. Specimen 5 is, thus, 50% stronger than specimen 8. Specimen 9 is also a top-angle and seat-angle connection requiring twenty-seven rivets. At the same rotation as the two other specimens, specimen 9 developed a resisting moment of 160 in-kips. All three specimens were used with the same size of beam—namely, a 12 I 31.8.

The split beam type of connection, if accompanied by reinforcing plates between the column flanges (as is the practice in welded girder-column connections) is practically a rigid connection and, in the opinion of the writer, should be so treated.

---

[26] "Elastic Properties of Riveted Connections," by J. Charles Rathbun, *Transactions*, ASCE, Vol. 101, 1936, p. 539.

[27] "Analysis of Building Frames with Semi-Rigid Connections," by Bruce Johnston and Edward H. Mount, *ibid.*, Vol. 107, 1942, p. 996, Fig. 1.

[28] "Elastic Properties of Riveted Connections," by J. Charles Rathbun, *ibid.*, Vol. 101, 1936, pp. 539–541.

The reason that the web-angle connection was chosen over the combination type for purposes of analyses is that the web-angle type is much easier to analyze. Analysis of the combination type may follow, but that for the web-angle connection will have paved the way.

Mr. Kavanagh poses the question as to how adequate the initial tangent is for practical design purposes. As was mentioned previously, using $\frac{1}{Z}$-values as computed by Eq. 22 is equivalent to designing the connection angles for a bending stress of 20.0 kips per sq in. This standard might be considered unnecessarily conservative by some designers. On the other hand, although the complete moment-rotation curves were available, in his examples, Mr. Rathbun chose the slope at the origin for his values of $1/Z$. Furthermore, an inspection of the Rathbun moment-rotation curves will show that the slope at the origin is approximately parallel to the average slope of the "set"[29] curves.

In answer to Mr. Kavanagh's concluding statement to the effect that the moment strengths indicated in Table 1 for the standard connections would quite probably be higher than will be allowed by combined moment and shear on the standard number of rivets usually provided in these connections, the writer wishes to refer him to the example in bracket design at the end of the paper. A standard connection was used, employing the standard number of the smaller sized rivets—namely, $\frac{3}{4}$-in. rivets. The bracket was designed to the full capacity of the rivets in combined moment and shear, and it was found that the connection angles were very much overstressed in moment.

[29] "Elastic Properties of Riveted Connections," by J. Charles Rathbun, *Transactions*, ASCE, Vol. 101, 1936, pp. 538–544 and 564.

AMERICAN SOCIETY OF CIVIL ENGINEERS

OFFICERS FOR 1950

PRESIDENT

ERNEST E. HOWARD

VICE-PRESIDENTS

Term expires January, 1951:

HENRY J. SHERMAN
ROBERT B. BROOKS

Term expires January, 1952:

FRED C. SCOBEY
ALBERT HAERTLEIN

DIRECTORS

Term expires January, 1951:

WILLIAM M. GRIFFIN
KIRBY SMITH
FRANCIS S. FRIEL
JULIAN HINDS
WEBSTER L. BENHAM
C. GLENN CAPPEL

Term expires January, 1952:

WALDO G. BOWMAN
MORRIS GOODKIND
HAROLD L. BLAKESLEE
PAUL L. HOLLAND
EDMUND FRIEDMAN
S. T. HARDING

Term expires January, 1953:

OTTO HOLDEN
FRANK L. WEAVER
GORDON H. BUTLER
LOUIS R. HOWSON
G. BROOKS EARNEST
WALTER J. RYAN
GEORGE W. LAMB

PAST-PRESIDENTS

Members of the Board

R. E. DOUGHERTY
FRANKLIN THOMAS

TREASURER
CHARLES E. TROUT

EXECUTIVE SECRETARY
WILLIAM N. CAREY

ASSISTANT TREASURER
GEORGE W. BURPEE

ASSISTANT SECRETARY
E. L. CHANDLER

PROCEEDINGS OF THE SOCIETY

SYDNEY WILMOT
Manager of Technical Publications

HAROLD T. LARSEN
Editor of Technical Publications

COMMITTEE ON PUBLICATIONS

WALDO G. BOWMAN

FRANCIS S. FRIEL
S. T. HARDING
OTTO HOLDEN
LOUIS R. HOWSON
KIRBY SMITH